

Constant Threshold Intersection Graphs of Orthodox Paths in Trees

Claudson Ferreira Bornstein¹ José Wilson Coura Pinto²
Dieter Rautenbach³ Jayme Luiz Szwarcfiter²

¹ DCC-IM, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil
cfb@cos.ufrj.br

² PESC-COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil
jwcoura, jayme@cos.ufrj.br

³ Institute of Optimization and Operations Research, Ulm University, Germany
dieter.rautenbach@uni-ulm.de

Abstract

A graph G belongs to the class $\text{ORTH}[h, s, t]$ for integers h , s , and t if there is a pair (T, \mathcal{S}) , where T is a tree of maximum degree at most h , and \mathcal{S} is a collection $(S_u)_{u \in V(G)}$ of subtrees S_u of maximum degree at most s of T , one for each vertex u of G , such that, for every vertex u of G , all leaves of S_u are also leaves of T , and, for every two distinct vertices u and v of G , the following three properties are equivalent:

- (i) u and v are adjacent.
- (ii) S_u and S_v have at least t vertices in common.
- (iii) S_u and S_v share a leaf of T .

The class $\text{ORTH}[h, s, t]$ was introduced by Jamison and Mulder.

Here we focus on the case $s = 2$, which is closely related to the well-known VPT and EPT graphs. We collect general properties of the graphs in $\text{ORTH}[h, 2, t]$, and provide a characterization in terms of tree layouts. Answering a question posed by Golumbic, Lipshteyn, and Stern, we show that $\text{ORTH}[h+1, 2, t] \setminus \text{ORTH}[h, 2, t]$ is non-empty for every $h \geq 3$ and $t \geq 3$. We derive decomposition properties, which lead to efficient recognition algorithms for the graphs in $\text{ORTH}[h, 2, 2]$ for every $h \geq 3$. Finally, we give a complete description of the graphs in $\text{ORTH}[3, 2, 2]$, and show that the graphs in $\text{ORTH}[3, 2, 3]$ are line graphs of planar graphs.

Keywords: Intersection graph; (h, s, t) -representation; orthodox (h, s, t) -representation; line graph; chordal graph

1 Introduction

Intersection graphs are a well studied topic [11, 17] and the intersection graphs of paths in trees have received special attention. In the present paper we study so-called orthodox representations with bounds on the maximum degree of the host tree as well as on the size of the intersections corresponding to adjacencies. Such representations were introduced by Jamison and Mulder [12, 13].

Before we give precise definitions and explain our own as well as related results, we collect some standard notation and terminology. We consider finite, undirected, and simple graphs as well as finite and undirected multigraphs, which are allowed to contain parallel edges and loops. A *clique* in G is a complete subgraph of G . For a tree T , let $\mathcal{L}(T)$ be the set of *leaves* of T , which are the vertices of T of degree at most 1. Let $L(H)$ be the *line graph* of some multigraph H , whose vertex set $V(L(H))$ is the edge set $E(H)$ of H , and in which two distinct vertices u and v of $L(H)$ are adjacent if and only if the edges u and v of H intersect. Two distinct vertices u and v of a graph G are *twins* in G if $N_G[u] = N_G[v]$, and, if G has no twins, then it is *twin-free*.

The following notions were formalized by Jamison and Mulder [12–14]. For positive integers h , s , and t , an (h, s, t) -*representation* of a graph G is a pair (T, \mathcal{S}) , where T is a tree of maximum degree at most h , and \mathcal{S} is a collection $(S_u)_{u \in V(G)}$ of subtrees S_u of maximum degree at most s of T , one for each vertex u of G , such that two distinct vertices u and v of G are adjacent if and only if S_u and S_v have at least t vertices in common. An (h, s, t) -representation (T, \mathcal{S}) of G with $\mathcal{S} = (S_u)_{u \in V(G)}$ is *orthodox* if, for every vertex u of G , all leaves of S_u are also leaves of T , and, for every two distinct vertices u and v of G , the following three properties are equivalent:

- (i) u and v are adjacent.
- (ii) S_u and S_v have at least t vertices in common.
- (iii) S_u and S_v share a leaf of T .

Let $[h, s, t]$ and $\text{ORTH}[h, s, t]$ be the classes of graphs that have an (h, s, t) -representation and an orthodox (h, s, t) -representation, respectively. If no upper bound on the maximum degree of the host T is imposed, we replace h with ∞ . Similarly, if no upper bound on the maximum degree of the subtrees in \mathcal{S} is imposed, we replace s with ∞ . Note that the classes $[h, s, t]$ and $\text{ORTH}[h, s, t]$ are hereditary, that is, closed under taking induced subgraphs. By iteratively removing irrelevant leaves of the host tree T of some orthodox (h, s, t) -representation (T, \mathcal{S}) , one may assume that every leaf of T is also a leaf of some tree in \mathcal{S} .

Using this terminology, Gavril’s famous result [6] states that the class of chordal graphs coincides with $[\infty, \infty, 1]$. Jamison and Mulder [13, 14] attribute to McMorris and Scheinerman [18] the insight that $[\infty, \infty, 1] = \text{ORTH}[3, 3, 1] = \text{ORTH}[3, 3, 2]$. In [13] they collect several properties of $[3, 3, 3]$ and $\text{ORTH}[3, 3, 3]$. The well studied vertex and edge intersection graphs of paths in trees [7–9], also known as VPT-graphs and EPT-graphs, coincide with $[\infty, 2, 1]$ and $[\infty, 2, 2]$, respectively. Golumbic and Jamison [8] have shown that deciding whether a given

graph belongs to $[3, 2, 1]$ is NP-complete. Alc3n, Gutierrez, and Mazzoleni [1] strengthened this result and generalized it for every $h \geq 3$. In [2] they study the forbidden induced subgraphs of $[h, 2, 1]$. Golumbic, Lipshteyn, and Stern [10] study the classes $[h, 2, t]$ and $\text{ORTH}[h, 2, t]$ in detail. In particular, they show that $\text{ORTH}[\infty, 2, 1] = \text{ORTH}[3, 2, 1] = \text{ORTH}[3, 2, 2]$, and that $\text{ORTH}[\infty, 2, 1]$ is a proper subclass of $\text{ORTH}[\infty, 2, 2]$. Furthermore, they ask whether $\text{ORTH}[\infty, 2, t]$ and $\text{ORTH}[3, 2, t]$ coincide.

In the present paper we study the classes $\text{ORTH}[h, 2, t]$.

For $h \leq 2$, these classes are rather simple. In fact, for every graph G in $\text{ORTH}[2, 2, 1]$, the vertex set $V(G)$ of G can be partitioned into three cliques A , B , and C , such that G contains all edges between A and B , all edges between B and C , but no edge between A and C , that is, the only connected twin-free graph in $\text{ORTH}[2, 2, 1]$ is P_3 . Furthermore, if $t \geq 2$, then the graphs in $\text{ORTH}[2, 2, t]$ consist of one clique and some isolated vertices. Hence, the smallest interesting value for h is 3.

In the second section we collect some general properties of the graphs in $\text{ORTH}[h, 2, t]$. Our main result is a characterization in terms of tree layouts whose precise definition will be given later. Using this characterization, we are able to answer the above-mentioned question by Golumbic, Lipshteyn, and Stern. We also derive some decomposition properties, which lead to efficient recognition algorithms for the graphs in $\text{ORTH}[h, 2, t]$ for every $h \geq 3$ and $t \in \{1, 2\}$; contrasting the above hardness results. In the third section we consider the classes $\text{ORTH}[3, 2, 2]$ and $\text{ORTH}[3, 2, 3]$ in more detail, and give a complete structural description of the first one. We conclude with some open problems motivated by our research.

2 General properties of $\text{ORTH}[h, 2, t]$

In this section we collect more general properties of the classes $\text{ORTH}[h, 2, t]$, and derive important structural consequences. Our first result closely ties these classes to line graphs.

Theorem 1 *Let (T, \mathcal{S}) be an orthodox $(h, 2, t)$ -representation of a graph G with $h \geq 3$ and $t \geq 1$ such that, for every leaf x of T , there is some vertex u of G with $x \in V(S_u)$.*

The graph G is the line graph of a multigraph H without loops, and, if G is twin-free, then H is a graph. Furthermore, if G is a connected twin-free graph of order at least 4, and H has no isolated vertices, then

- *H is unique up to isomorphism,*
- *there is a bijection $\phi : V(H) \rightarrow \mathcal{L}(T)$, and*
- *two distinct vertices x and y of H are adjacent in H if and only if \mathcal{S} contains the path in T between $\phi(x)$ and $\phi(y)$.*

Proof: Let $\mathcal{S} = (S_u)_{u \in V(G)}$. By definition, for every vertex u of G , the subtree S_u is a path between leaves of T .

Suppose that, for some vertex u of G , the path S_u consists of only a single leaf, say x , of T . Let y be the neighbor of x in T . Let T' arise from T by adding the two new vertices x' and x'' as well as the two new edges xx' and xx'' . Now,

- replacing S_v by the path $x'xx''$ for every vertex v of G for which S_v consists only of x , and,
- extending S_v by the edge xx' for every vertex v of G for which S_v contains the edge xy of T

yields an alternative orthodox $(h, 2, t)$ -representation of G using T' as host tree. Possibly applying this transformation several times, we may assume that every path in \mathcal{S} has positive length.

By definition, for every leaf x of T , the set $C_x = \{u \in V(G) : x \in V(S_u)\}$ is a clique in G . Since, for every vertex u of G , the subtree S_u is a path between two distinct leaves of T , every vertex of G belongs to exactly two of the cliques in the collection $(C_x)_{x \in \mathcal{L}(T)}$. Furthermore, for every edge uv of G , the two subtrees S_u and S_v share a leaf, say x , of T , which implies that u and v both belong to C_x . By results of Krausz [15] and of Bermond and Meyer [3], this implies that G is the line graph of some multigraph H without loops. Since parallel edges in H correspond to twins in G , if G is twin-free, then H is a graph.

Now, let G be twin-free, connected, and of order at least 4. By a result of Whitney [19], the graph H is uniquely determined. Let H' be the graph with $V(H') = \mathcal{L}(T)$ in which two distinct vertices x and y are adjacent if and only if \mathcal{S} contains the path in T between x and y . Since every leaf of T is also a leaf of some S_u , the graph H' has no isolated vertex. By definition, the graph G is isomorphic to $L(H')$, and, by Whitney's result, the graphs H' and H are isomorphic. \square

If the graph G' arises from a graph G by identifying all pairs of twins in G , then G' is twin-free. Furthermore, it follows easily from the definition that G is in $\text{ORTH}[h, 2, t]$ if and only if G' is.

For $i \in [2]$, let (T_i, \mathcal{S}_i) be an orthodox $(h, 2, t)$ -representation of a graph G_i for some $h \geq 3$ and $t \geq 1$, where G_1 and G_2 as well as T_1 and T_2 are disjoint. Let T be the tree that arises from the disjoint union of T_1 and T_2 by subdividing one edge of T_i containing a leaf of T_i with a new vertex t_i for $i \in [2]$, and adding the edge t_1t_2 . Since $h \geq 3$, applying the same subdivisions to the trees in $\mathcal{S}_1 \cup \mathcal{S}_2$, it follows that there is an orthodox $(h, 2, t)$ -representation of the disjoint union of G_1 and G_2 using T as host tree.

In view of these observations and Theorem 1, in order to understand the classes $\text{ORTH}[h, 2, t]$, it suffices to consider connected twin-free line graphs G of order at least 4 as well as connected graphs H with $L(H) = G$.

The next result is our central characterization of the graphs in $\text{ORTH}[h, 2, t]$.

Theorem 2 *Let G be a connected twin-free line graph of order at least 4, and let H be a connected graph with $L(H) = G$.*

The graph G is in $\text{ORTH}[h, 2, t]$ for some $h \geq 3$ and $t \geq 1$ if and only if there is a tree T whose internal vertices all have degree at most h such that $V(H) = \mathcal{L}(T)$, and, for every two independent edges xy and $x'y'$ of H , the two paths in T between x and y and between x' and y' share at most $t - 1$ vertices.

Proof: Let G be in $\text{ORTH}[h, 2, t]$. As shown in the proof of Theorem 1, there is an orthodox $(h, 2, t)$ -representation (T, \mathcal{S}) of G such that every path in $\mathcal{S} = (S_u)_{u \in V(G)}$ has positive length. By definition, all internal vertices of T have degree at most h . By Theorem 1, we may assume that $V(H) = \mathcal{L}(T)$, and that two distinct vertices x and y of H are adjacent in H if and only if \mathcal{S} contains the path in T between x and y . If xy and $x'y'$ are two independent edges of H , then there are two vertices u and v in G such that S_u is the path in T between x and y and S_v is the path in T between x' and y' . Since S_u and S_v share no leaf, the two vertices u and v are not adjacent in G , which implies that S_u and S_v share at most $t - 1$ vertices.

Now, let T be as in the statement. Let T' arise from T by subdividing each edge incident with a leaf of T exactly $t - 2$ times. Note that T' still has maximum degree at most h , and that, for every two independent edges xy and $x'y'$ of H , the two paths in T' between x and y and between x' and y' share at most $t - 1$ vertices. Let $\mathcal{S} = (S_{xy})_{xy \in E(H)}$, where S_{xy} is the path in T' between the leaves x and y of T' . Let u and v be two distinct vertices of G . Let $u = xy$ and $v = x'y'$, where xy and $x'y'$ are the corresponding edges of H . Now, u and v are adjacent in G if and only if the edges xy and $x'y'$ are not independent if and only if S_{xy} and $S_{x'y'}$ share a leaf of T' if and only if S_{xy} and $S_{x'y'}$ have at least t vertices in common. This implies that (T', \mathcal{S}) is an orthodox $(h, 2, t)$ -representation of G . \square

The relation between H and T in the previous theorem is crucial for all our considerations, and we introduce some corresponding terminology.

If H is a graph and T is a tree such that

- the maximum degree of T is at most h ,
- $V(H) = \mathcal{L}(T)$, and,
- for every two independent edges xy and $x'y'$ of H , the two paths in T between x and y and between x' and y' share at most $t - 1$ vertices

for some integers $h \geq 3$ and $t \geq 1$, then T is an (h, t) -tree layout of H .

As we see now, vertices of degree 2 are not essential within tree layouts.

Corollary 3 *Let G be a connected twin-free line graph of order at least 4, and let H be a connected graph with $L(H) = G$.*

The graph G is in $\text{ORTH}[3, 2, t]$ for some $t \geq 1$ if and only if there is a tree T whose internal vertices all have degree exactly 3 such that $V(H) = \mathcal{L}(T)$, and, for every two independent edges xy and $x'y'$ of H , the two paths in T between x and y and between x' and y' share at most $t - 1$ vertices.

Proof: In view of Theorem 2, it suffices to argue that no internal vertex of T needs degree 2. Therefore, let T be an (h, t) -tree layout of H . If some vertex b of T has exactly two neighbors a and c within T , then it is easy to see that $T' = T - b + ac$ is still an (h, t) -tree layout of H . Iterating this transformation, it is possible to eliminate all internal vertices of T that are of degree 2. \square

Note that every tree with n leaves whose internal vertices all have degree 3, has order exactly $2n - 2$. Therefore, if G , H , and T are as in the statement of Corollary 3, then subdividing the edges incident with leaves of T at most $t - 2$ times, and defining \mathcal{S} as in the proof of Theorem 2 yields an orthodox $(3, 2, t)$ -representation of G whose underlying tree has order between $2n(H) - 2$ and $tn(H) - 2$. Hence, by Theorem 2, the minimum order of an underlying tree in any orthodox $(3, 2, t)$ -representation of G lies between these two bounds.

Our next goal is to answer the question posed by Golumbic, Lipshteyn, and Stern [10]. In fact, we show that line graphs of complete graphs of suitable orders distinguish the classes $\text{ORTH}[h, 2, t]$ for different values of h . The next lemma is a simple exercise, and we include the proof for completeness.

Lemma 4 *Let h and t be integers with $h \geq 3$ and $t \geq 3$.*

If T is a tree of maximum degree at most h such that every two leaves of T have distance at most t , then

$$|\mathcal{L}(T)| \leq \begin{cases} 2(h-1)^{\binom{t-1}{2}} & , \text{ if } t \text{ is odd, and} \\ h(h-1)^{\binom{t-1}{2}} & , \text{ if } t \text{ is even.} \end{cases}$$

Furthermore, these bounds are tight for all considered values of h and t .

Proof: In view of the desired bound, we may assume that T has two leaves at distance t . Let $x(0) \dots x(t)$ be a path between two such leaves.

First, let t be odd. Rooting the two components of $T - x\left(\frac{t-1}{2}\right) x\left(\frac{t-1}{2} + 1\right)$ in the two vertices $x\left(\frac{t-1}{2}\right)$ and $x\left(\frac{t-1}{2} + 1\right)$ yields rooted $(h-1)$ -ary trees T_1 and T_2 of depth $\frac{t-1}{2}$. Clearly,

$$|\mathcal{L}(T)| = |\mathcal{L}(T_1)| + |\mathcal{L}(T_2)| \leq (h-1)^{\binom{t-1}{2}} + (h-1)^{\binom{t-1}{2}}$$

with equality if and only if T_1 and T_2 are full $(h-1)$ -ary trees of depth $\frac{t-1}{2}$.

Next, let t be even. Rooting the two components of $T - x\left(\frac{t}{2}\right) x\left(\frac{t}{2} + 1\right)$ in the two vertices $x\left(\frac{t}{2}\right)$ and $x\left(\frac{t}{2} + 1\right)$ yields rooted $(h-1)$ -ary trees T_1 of depth $\frac{t}{2}$ and T_2 of depth $\frac{t}{2} - 1$. Clearly,

$$|\mathcal{L}(T)| = |\mathcal{L}(T_1)| + |\mathcal{L}(T_2)| \leq (h-1)^{\binom{t}{2}} + (h-1)^{\binom{t}{2}-1} = h(h-1)^{\binom{t}{2}-1}$$

with equality if and only if T_1 and T_2 are full $(h-1)$ -ary trees of depths $\frac{t}{2}$ and $\frac{t}{2} - 1$, respectively. \square

The following result answers the question posed by Golumbic, Lipshteyn, and Stern [10] mentioned in the introduction. For two integers p and q , let $[p, q]$ denote the set of integers at least p and at most q .

Theorem 5 *Let h and t be integers with $h \geq 3$ and $t \geq 3$.*

If t is odd, then

$$\left\{ n \in \mathbb{N} : L(K_n) \in \text{ORTH}[h+1, 2, t] \setminus \text{ORTH}[h, 2, t] \right\} = \left[2(h-1)^{\binom{t-1}{2}} + 1, 2h^{\binom{t-1}{2}} \right],$$

and, if t is even, then

$$\left\{ n \in \mathbb{N} : L(K_n) \in \text{ORTH}[h+1, 2, t] \setminus \text{ORTH}[h, 2, t] \right\} = \left[h(h-1)^{\binom{t}{2}-1} + 1, (h+1)h^{\binom{t}{2}-1} \right].$$

Proof: We only give details for odd t , because the proof for even t is analogous. Since the considered classes are hereditary, it suffices to show that

- (i) $L\left(K_{2h^{\binom{t-1}{2}}}\right) \in \text{ORTH}[h+1, 2, t]$, and
- (ii) $L\left(K_{2(h-1)^{\binom{t-1}{2}} + 1}\right) \notin \text{ORTH}[h, 2, t]$.

By Lemma 4, there is a tree of maximum degree at most $h+1$ with $2h^{\binom{t-1}{2}}$ leaves such that every two leaves have distance at most t . By Theorem 2, this implies (i).

Now, suppose that (ii) does not hold. Again by Theorem 2, there is a tree T of maximum degree at most h with $2(h-1)^{\binom{t-1}{2}}$ leaves such that, for every four distinct leaves u_1, v_1, u_2 , and v_2 , the two paths in T between u_1 and v_1 and between u_2 and v_2 share at most $t-1$ vertices. Let u_1 and v_1 be two leaves of T with maximum distance ℓ . We assume that T is chosen such that ℓ is as small as possible. Let u'_1 and v'_1 be the two neighbors of u_1 and v_1 , respectively. Clearly, the vertices u'_1 and v'_1 are distinct. By the choice of T , the vertex u'_1 is adjacent to a leaf u_2 distinct from u_1 , and the vertex v'_1 is adjacent to a leaf v_2 distinct from v_1 . Considering the two independent edges u_1v_1 and u_2v_2 implies that $\ell \leq t$. By Lemma 4, this implies the contradiction that T has at most $2(h-1)^{\binom{t-1}{2}}$ leaves. \square

Next, we show that all subgraphs of the essential graphs H with $L(H) \in \text{ORTH}[h, 2, t]$ have balanced separations of bounded order.

Theorem 6 *Let G be a connected twin-free line graph of order at least 4, and let H be a connected graph with $L(H) = G$. Let G be in $\text{ORTH}[h, 2, t]$ for some $h \geq 3$ and $t \geq 1$.*

If H' is a subgraph of H of order at least 2, then there is a set X of vertices of H' , and a partition of $V(H')$ into two sets A and B such that

- (i) $\frac{1}{h}n(H') \leq |A|, |B| \leq \frac{h-1}{h}n(H')$,
- (ii) $|X| \leq \max\{1, (h-1)^{(t-2)}\}$, and
- (iii) H' contains no edge between $A \setminus X$ and $B \setminus X$.

Furthermore, given H' , the sets X , A , and B can be found in polynomial time.

Proof: Let T be as in Theorem 2, that is, the tree T is an (h, t) -tree layout of H . Clearly, iteratively removing leaves of T that are not vertices of H' yields an (h, t) -tree layout T' of H' .

If $t = 1$, u is some vertex of T' , Y is the set of leaves of some components of $T' - u$, and Z is the set of leaves of the remaining components of $T' - u$, then the properties of a $(h, 1)$ -tree layout imply the existence of a vertex x in $Y \cup Z$ such that all edges of H' between Y and Z are incident with x , because otherwise H' contains two independent edges between Y and Z such that the corresponding paths in T' share u . Similarly, if $t \geq 2$, $P : u_1 \dots u_t$ is a path of order t in T' , Y is the set of leaves of T' that lie in the same component of $T' - E(P)$ as u_1 , and Z is the set of leaves of T' that lie in the same component of $T' - E(P)$ as u_t , then there is a vertex x in $Y \cup Z$ such that all edges of H' between Y and Z are incident with x .

Let r be any internal vertex of T' , and root T' in r . Let a be a vertex of T' of maximum distance from r such that at least $\frac{1}{h}n(H')$ descendants of a are leaves. Since T' has $n(H')$ leaves and r has degree at most h , the vertex a is not r . Since a has at most $h - 1$ children, at most $\frac{h-1}{h}n(H')$ descendants of a are leaves. Let b be the parent of a . Let A be the set of leaves of T' that lie in the same component of $T' - ab$ as a , and let B be the set of leaves of T' that lie in the same component of $T' - ab$ as b . Clearly, (i) holds.

If $t = 1$, then the above observation implies the existence of a single vertex x such that all edges of H' between A and B are incident with x , and, (ii) and (iii) follow. Now, let $t \geq 2$. Rooting the component T'_b of $T' - ab$ that contains b in the vertex b , and considering all leaves of T'_b at depth at most $t - 2$ as well as all non-leaf vertices of T'_b at depth exactly $t - 2$, it follows that B can be partitioned into $k \leq (h - 1)^{t-2}$ sets B_1, \dots, B_k such that, for every $i \in [k]$, if B_i contains not only one vertex, then there is a path P_i of order t such that every path in T' between a leaf in A and a leaf in B_i has P_i as a subpath. If B_i contains only one vertex, then trivially all edges of H' between A and B_i are incident with only one vertex in $A \cup B_i$. If B_i contains not only one vertex, then the above observation also implies that all edges of H' between A and B_i are incident with only one vertex in $A \cup B_i$. Altogether, it follows that there is a set X of at most k vertices of H' such that all edges between A and $B = B_1 \cup \dots \cup B_k$ are incident with a vertex in X , and, (ii) and (iii) follow.

It remains to explain, how to determine suitable sets X , A , and B efficiently given H' . Therefore, let $p = \max \{1, (h - 1)^{(t-2)}\}$. If $p \geq \frac{h-1}{h}n(H')$, then choosing A as any set of $\lceil \frac{1}{h}n(H') \rceil$ vertices of H' , and $B = X$ as $V(H') \setminus A$ satisfies (i), (ii), and (iii). Now, let $p < \frac{h-1}{h}n(H')$. For any specific set X of p vertices of H' , we explain how to decide whether sets A and B with (i), (ii), and (iii) exist. Therefore, let X be such a set. Let K_1, \dots, K_ℓ be the components of $H' - X$. If some component K_i with $i \in [\ell]$ has order more than $\frac{h-1}{h}n(H')$, then the desired sets A and B do not exist. If some component K_i with $i \in [\ell]$ has order between $\frac{1}{h}n(H')$ and $\frac{h-1}{h}n(H')$, then choosing $A = V(K_i)$ and $B = V(H') \setminus A$ has the desired properties. Finally, if all components K_i with $i \in [\ell]$ have order less than $\frac{1}{h}n(H')$, then forming a union of suitably many of their vertex sets yields a set A of order between $\frac{1}{h}n(H')$ and $\frac{h-1}{h}n(H')$, and defining B as above leads to sets with the desired properties. Altogether, considering the $O(n(H')^p)$ many choices for X , suitable sets can be determined in polynomial time, which

completes the proof. \square

The following immediate consequence of Theorem 6 might be useful in order to devise efficient recognition algorithms for the graphs in $\text{ORTH}[h, 2, t]$.

Corollary 7 *If G is a connected twin-free line graph of order at least 4 in $\text{ORTH}[h, 2, t]$ for some $h \geq 3$ and $t \geq 1$, and H is a connected graph with $L(H) = G$, then the treewidth of H is bounded as a function of h and t .*

Proof: This follows easily from Theorem 6 and a result of Dvořák and Norin [5]. \square

Our final goal in this section are efficient recognition algorithms for the graphs in $\text{ORTH}[h, 2, 1]$ and $\text{ORTH}[h, 2, 2]$ based on Theorem 6.

Lemma 8 *Let H be a graph. Let the sets A and B partition $V(H)$. Let a in A and let b in B be such that every edge of H between A and B is incident with a , and b is a neighbor of a .*

The graph H has an (h, t) -tree layout for some $h \geq 3$ and $t \geq 1$ if and only if the two graphs $H_A = H[A \cup \{b\}]$ and $H_B = H[B \cup \{a\}]$ have (h, t) -tree layouts.

Proof: If H has an (h, t) -tree layout, then so does every induced subgraph, which implies the necessity. For the sufficiency, assume that T_A and T_B are (h, t) -tree layouts of H_A and H_B , respectively. Note that T_A and T_B share exactly the two vertices a and b . If the tree T arises from the disjoint union of T_A and T_B , where we distinguish the two copies of a and b within T_A and T_B , by adding an edge between the copy of b in T_A and the copy of a in T_B , then it follows easily that T is an (h, t) -tree layout of H , which completes the proof. \square

Corollary 9 *For every two integers $h \geq 3$ and $t \in \{1, 2\}$, the graphs in $\text{ORTH}[h, 2, t]$ can be recognized in polynomial time.*

Proof: Let G be a given graph for which we want to decide whether it belongs to $\text{ORTH}[h, 2, t]$. As observed after Theorem 1, we may assume that G is a connected twin-free line graph of order at least 4. Using for instance the algorithms in [4], we can efficiently determine the unique connected graph H with $L(H) = G$. Clearly, $n(H) \leq m(H) + 1 = n(G) + 1$. By Theorem 2, we need to decide whether H has an (h, t) -tree layout.

Note that, for $t \in \{1, 2\}$, the set X in Theorem 6 contains at most one vertex. Furthermore, sets X , A , and B with (i), (ii), and (iii) can be found efficiently for every subgraph H' of H of order at least 2. Note that the graphs H_A and H_B considered in Lemma 8 have orders $|A| + 1$ and $|B| + 1$, respectively. Let n_0 be such that $\frac{h-1}{h}n + 1 \leq \frac{h}{h+1}n$ for $n \geq n_0$. Note that, if H has order at least n_0 , then H_A and H_B both have orders at most $\frac{h}{h+1}n(H)$. Therefore, iteratively applying Theorem 6 yields $k \leq 2^{\lceil \log_{(h+1)/h}(n/n_0) \rceil}$ many graphs H_1, \dots, H_k , each of order at most n_0 , such that H has an (h, t) -tree layout if and only if each H_i has an (h, t) -tree layout for every $i \in [k]$. Clearly, testing this property for these polynomially many graphs of bounded order can be done in polynomial time. \square

3 ORTH[3, 2, 2] and ORTH[3, 2, 3]

This section collects more specific properties of the classes $\text{ORTH}[3, 2, 2]$ and $\text{ORTH}[3, 2, 3]$. As shown by Golumbic, Lipshteyn, and Stern [10], we have $\text{ORTH}[\infty, 2, 1] = \text{ORTH}[3, 2, 1] = \text{ORTH}[3, 2, 2]$, which implies $\text{ORTH}[h, 2, 1] = \text{ORTH}[3, 2, 2]$ for every $h \geq 3$.

Theorem 10 *Let G be a connected twin-free line graph of order at least 4, and let H be a connected graph with $L(H) = G$.*

The graph G is in $\text{ORTH}[3, 2, 2]$ if and only if all blocks of H are of order at most 3.

Proof: In order to show the necessity, we assume that G is in $\text{ORTH}[3, 2, 1] = \text{ORTH}[3, 2, 2]$ but that some block of H has order at least 4. This implies that H has some cycle C of length ℓ at least 4 as a subgraph. Hence, G contains the induced cycle $L(C)$ of length ℓ . Nevertheless, by Gavril's result [6], the graphs in $\text{ORTH}[3, 2, 1]$ are chordal, which is a contradiction.

In order to show the sufficiency, we assume that all blocks of H are of order at most 3. Since K_3 has a $(3, 1)$ -tree layout, it follows easily by an inductive argument similar to the proof of Lemma 8 that H has a $(3, 1)$ -tree layout. Suppose, for instance, that H arises from a smaller graph H' containing a vertex u by adding two vertices v and w , and three new edges uv , uw , and vw , that is, H arises from H' by attaching one new K_3 block. If T' is a $(3, 1)$ -tree layout of H' , then subdividing the edge of T' incident with u by a new vertex x , adding three more vertices y , v , and w , and adding three more edges xy , vy , and wy yields a $(3, 1)$ -tree layout of H . By Theorem 2, this implies that G is in $\text{ORTH}[3, 2, 1] = \text{ORTH}[3, 2, 2]$. \square

Lemma 11 *If e and f are two independent edges of K_5 , H is a subdivision of $K_5 - \{e, f\}$, and $G = L(H)$, then G is not in $\text{ORTH}[3, 2, 3]$.*

Proof: Let $H_0 = K_5 - \{e, f\}$. We denote the five vertices of H_0 by u_1, \dots, u_5 without specifying which two edges are missing. Note that G is a connected twin-free line graph of order at least 4. For a contradiction, suppose that G is in $\text{ORTH}[3, 2, 3]$. Let T be as in Corollary 3 for $t = 3$. Let r be any internal vertex of T , and root T in r . Let s be a vertex of T of maximum distance from r such that at least two descendants, say u_1 and u_2 , of s within T are vertices of H_0 . Since every internal vertex of T has degree 3, exactly two descendants of s are vertices of H_0 , which implies that s is not r . Let t be the parent of s . Let s' and s'' be the two neighbors of t distinct from s . Let S , S' , and S'' be the vertex sets of the three components of $T - t$ that contain s , s' , and s'' , respectively.

By the pigeonhole principle and by symmetry, we may assume that u_4 and u_5 lie in S' , and that u_1u_4 and u_2u_5 are edges of H_0 . Note that every edge uv of H_0 corresponds to a path $P(uv)$ in H between u and v whose internal vertices are all of degree 2, and that the edges of this path correspond to leaf to leaf paths in T such that, for independent edges, the corresponding paths share at most 2 vertices.

Since the two paths $P(u_1u_4)$ and $P(u_2u_5)$ are between a vertex in S and a vertex in S' , each contains

- an edge between S and S' , or
- an edge between S and S'' as well as an edge between S' and S'' .

Since any edge of $P(u_1u_4)$ is disjoint from any edge of $P(u_2u_5)$, the structure of T implies that we may assume that $P(u_1u_4)$ contains an edge e_1 between S and S' , and that $P(u_2u_5)$ contains an edge e_2 between S and S'' as well as an edge e'_2 between S' and S'' . If u_1 is adjacent to u_5 in H_0 , then, in view of e_2 , the path $P(u_1u_5)$ contains an edge between S and S' , and, if u_2 is adjacent to u_4 in H_0 , then, in view of e'_2 , the path $P(u_2u_4)$ contains an edge between S and S' . Since every edge of $P(u_1u_5)$ is disjoint from every edge of $P(u_2u_4)$, this implies that u_1u_5 or u_2u_4 is one of the two non-edges of H_0 . If u_3 is in S'' , then, in view of e_2 and e'_2 , the vertex u_3 can not be adjacent to u_1 or u_4 in H_0 , which is a contradiction. Hence, u_3 is in S' . In view of e_1 and e'_2 , the vertex u_3 is not adjacent to u_2 in H_0 . Together with our earlier observation, this implies that the two missing edges of H_0 are exactly u_1u_5 and u_2u_3 . In view of e_2 , the path $P(u_1u_3)$ contains an edge e_3 between S and S' . In view of e'_2 , the path $P(u_2u_4)$ contains an edge e_4 between S and S' . Now, the two paths in T between the endpoints of e_3 as well as between the endpoints of e_4 share three vertices s , t , and s' , which is a contradiction. \square

The previous lemma has a suitable generalization for larger values of h than 3. A similar proof also shows that $L(K_{2,5})$ does not lie in $\text{ORTH}[3, 2, 3]$.

Lemma 12 *If H is a subdivision of $K_{3,3}$ and $G = L(H)$, then G is not in $\text{ORTH}[3, 2, 3]$.*

Proof: Let the two partite sets of $K_{3,3}$ be $\{u_1, u_2, u_3\}$ and $\{u'_1, u'_2, u'_3\}$. Note that G is a connected twin-free line graph of order at least 4. For a contradiction, suppose that G is in $\text{ORTH}[3, 2, 3]$. Let T be as in Corollary 3 for $t = 3$. Let r be any internal vertex of T , and root T in r . Let s be a vertex of T of maximum distance from r such that at least two descendants of s within T are vertices of $K_{3,3}$. Again, exactly two descendants of s are vertices of $K_{3,3}$. Let t, s', s'', S, S' , and S'' be as in the proof of Lemma 11.

We consider some cases concerning the two vertices of $K_{3,3}$ in S as well as the distribution of the remaining four vertices $K_{3,3}$ within S' and S'' .

Case 1 $u_1, u_2 \in S$.

By symmetry, we may assume that $u'_1, u'_2 \in S'$. Arguing as in the proof of Lemma 11, we may assume, by symmetry, that $P(u_1u'_1)$ contains an edge e_1 between S and S'' as well as an edge e'_1 between S' and S'' , and that $P(u_2u'_2)$ contain an edge e_2 between S and S' . In view of e_1 , the path $P(u_2u'_3)$ contains no edge between S and S'' . In view of e'_1 , the path $P(u_2u'_3)$ contains no edge between S' and S'' . This implies that $u'_3 \in S'$. In view of e_2 , the path $P(u_1u'_3)$ contains no edge between S and S' . In view of e'_1 , the path $P(u_1u'_3)$ contains no edge between S' and S'' . This implies a contradiction.

In view of Case 1, we may assume that the two vertices of $K_{3,3}$ in S belong to different partite sets, that is, by symmetry, $u_1, u'_1 \in S$.

Case 2 $u_1, u'_1 \in S, u_2, u'_2 \in S', \text{ and } u_3, u'_3 \in S''$.

By symmetry, we may assume that $P(u_1u'_2)$ contains an edge e_1 between S and S'' as well as an edge e'_1 between S' and S'' , and that $P(u'_1u_2)$ contains an edge e_2 between S and S' . In view of e_1 , the path $P(u_2u'_3)$ contains no edge between S and S'' . In view of e'_1 , the path $P(u_2u'_3)$ contains no edge between S' and S'' . This implies a contradiction.

Case 3 $u_1, u'_1 \in S$ and $u_2, u'_2, u_3, u'_3 \in S'$.

By symmetry, we may assume that $P(u_1u'_2)$ contains an edge e_1 between S and S'' as well as an edge e'_1 between S' and S'' , and that $P(u'_1u_2)$ contains an edge e_2 between S and S' . In view of e_2 , the path $P(u_1u'_3)$ contains no edge between S and S' . In view of e'_1 , the path $P(u_1u'_3)$ contains no edge between S' and S'' . This implies a contradiction.

In view of Cases 2 and 3, we may assume, by symmetry, that $u_2 \in S'$ and $u'_2 \in S''$. If $u_3 \in S'$ and $u'_3 \in S''$, then we can argue as in Case 1. If $u'_3 \in S'$ and $u_3 \in S''$, then we can argue as in Case 2. Hence, by symmetry, it suffices to consider the following final case.

Case 4 $u_1, u'_1 \in S$, $u_2, u_3, u'_3 \in S'$, and $u'_2 \in S''$.

By symmetry, we may assume that $P(u_1u'_3)$ contains an edge e_1 between S and S'' as well as an edge e'_1 between S' and S'' , and that $P(u'_1u_3)$ contains an edge e_2 between S and S' . In view of e_2 , the path $P(u_2u'_2)$ contains no edge between S and S' . In view of e'_1 , the path $P(u_2u'_2)$ contains no edge between S' and S'' . This implies a contradiction, and completes the proof. \square

Theorem 13 *If a connected twin-free graph G of order at least 4 is in $\text{ORTH}[3, 2, 3]$, then G is the line graph of a planar graph.*

Proof: By Theorem 1, there is a unique connected graph H' with $G = L(H')$. For a contradiction, suppose that H' is not planar. By a result of Kuratowski [16], the graph H' has a subgraph H that is a subdivision of a graph H_0 such that H_0 is either K_5 or $K_{3,3}$. Since $L(H)$ is an induced subgraph of $L(H')$, the graph $L(H)$ is in $\text{ORTH}[3, 2, 3]$. Now, if H_0 is K_5 or $K_{3,3}$, then Lemma 11 or Lemma 12 implies the contradiction that $L(H)$ is not in $\text{ORTH}[3, 2, 3]$, respectively. \square

Since Lemma 11 actually concerns subdivisions of $K_5 - e$, the containment in Theorem 13 is proper.

4 Conclusion

The most natural open problems concern the structure of the graphs in the classes $\text{ORTH}[3, 2, 3]$ and $\text{ORTH}[h, 2, 2]$ for $h \geq 4$. For $\text{ORTH}[3, 2, 3]$, the complexity of the recognition is unknown. In view of Corollary 7, efficient recognition algorithms for all classes $\text{ORTH}[h, 2, t]$ seem possible. Our results should have further algorithmic consequences. If, for example, G and H are as in Theorem 2, then the chromatic number of G is either the maximum degree of H or one more, and, by Corollary 7, these two cases can be distinguished efficiently.

References

- [1] L. Alcón, M. Gutierrez, M.P. Mazzoleni, Recognizing vertex intersection graphs of paths on bounded degree trees, *Discrete Applied Mathematics* 162 (2014) 70-77.
- [2] L. Alcón, M. Gutierrez, M.P. Mazzoleni, Characterizing $[h, 2, 1]$ graphs by minimal forbidden induced subgraphs, *arXiv:1307.2139v1*.
- [3] J.C. Bermond, J.C. Meyer, Graphe représentatif des aretes d'un multigraphe, *Journal de Mathématiques Pures et Appliquées, Neuvième Série* 52 (1973) 299-308.
- [4] D.G. Degiorgi, K. Simon, A dynamic algorithm for line graph recognition, *Lecture Notes in Computer Science* 1017 (1995) 37-48.
- [5] Z. Dvořák, S. Norin, Treewidth of graphs with balanced separations, *arXiv:1408.3869v1*.
- [6] F. Gavril, The intersection graphs of subtrees in a tree are exactly the chordal graphs, *Journal of Combinatorial Theory, Series B* 16 (1974) 47-56.
- [7] F. Gavril, A recognition algorithm for the intersection graphs of paths in trees, *Discrete Mathematics* 23 (1978) 211-227.
- [8] M.C. Golumbic, R.E. Jamison, Edge and vertex intersection of paths in a tree, *Discrete Mathematics* 55 (1985) 151-159.
- [9] M.C. Golumbic, R.E. Jamison, The edge intersection graphs of paths in a tree, *Journal of Combinatorial Theory, Series B* 38 (1985) 8-22.
- [10] M.C. Golumbic, M. Lipshteyn, M. Stern, Equivalences and the complete hierarchy of intersection graphs of paths in a tree, *Discrete Applied Mathematics* 156 (2008) 3203-3215.
- [11] M.C. Golumbic, A.N. Trenk, *Tolerance graphs*, Cambridge University Press, 2004.
- [12] R.E. Jamison, H.M. Mulder, Constant tolerance representations of graphs in trees, *Congressus Numerantium* 143 (2000) 175-192.
- [13] R.E. Jamison, H.M. Mulder, Tolerance intersection graphs on binary trees with constant tolerance 3, *Discrete Mathematics* 215 (2000) 115-131.
- [14] R.E. Jamison, H.M. Mulder, Constant tolerance intersection graphs of subtrees of a tree, *Discrete Mathematics* 290 (2005) 27-46.
- [15] J. Krausz, Démonstration nouvelle d'une théorème de Whitney sur les réseaux, *Matematikai és Fizikai Lapok* 50 (1943) 75-89.
- [16] C. Kuratowski, Sur le problème des courbes gauches en topologie, *Fundamenta Mathematicae* 15 (1930) 271-283.

- [17] T.A. McKee, F.R. McMorris, Topics in intersection graph theory, SIAM Monographs on Discrete Mathematics and Applications, 1999.
- [18] F.R. McMorris, E. Scheinerman, Connectivity threshold for random chordal graphs, *Graphs and Combinatorics* 7 (1991) 177-181.
- [19] H. Whitney, Congruent graphs and the connectivity of graphs, *American Journal of Mathematics* 54 (1932) 150-168.